# Compressed sensing using Generative Models.

**Theorem 1.1:** Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be a generative model from a $d$-layer neural network with ReLU activations. Let $A \in \mathbb{R}^{m \times n}$ be a random Gaussian matrix for $m = O(kd \log n)$ measurements. $A_{ij} \overset{iid}{\sim} N(0, 1/m)$. For any $x^* \in \mathbb{R}^n$, and $y = Ax^* + \eta$, let $\hat{z}$ be such that

$$\|y - AG(\hat{z})\| \leq \min_{z \in \mathbb{R}^k} \|y - AG(z)\| + \epsilon.$$

Then, with probability $1 - e^{-\Omega(m)}$,

$$\|G(\hat{z}) - x^*\| \leq 6 \min_{z^* \in \mathbb{R}^k} \|G(z) - x^*\| + 3\|\eta\| + 2\epsilon.$$

Proof: From lemma 4.1, we know that if $m = \Omega((kd \log c)/\alpha^2)$, $(A_{ij}) \overset{iid}{\sim} (0, 1/m)$ satisfies S-REC$(G(\mathbb{R}^k), 1-\alpha, 0)$ w.p. $1 - e^{-\Omega(\alpha^2 m)}$. Now, applying lemma 4.3, with $\hat{x} = G(\hat{z})$, $S = G(\mathbb{R}^k)$

$$\|G(\hat{z}) - x^*\| \leq \left(\frac{4}{1-\alpha} + 1\right) \min_{z \in \mathbb{R}^k} \|G(z) - x^*\| + \frac{1}{1-\alpha}\left(2\|\eta\| + \epsilon\right)$$

choosing $\alpha = 1/5$,

$$\|G(\hat{z}) - x^*\| \leq 6 \min_{z \in \mathbb{R}^k} \|G(z) - x^*\| + 2.5\|\eta\| + \frac{5}{4}\epsilon$$

$\boxtimes$

Theorem 1.2: Let $G : \mathbb{R}^k \to \mathbb{R}^n$ be an L-Lipschitz function. Let $A \in \mathbb{R}^{m \times n}$ be a random Gaussian matrix for $m = O(k \log Lr/\delta)$ s.t $(A_{ij}) \overset{iid}{\sim} N(0, 1/m)$. For any $x^* \in \mathbb{R}^n$, $y = Ax^* + \eta$, let $\hat{z}$ be output of algo. s.t

$$\|y - AG(\hat{z})\| \leq \min_{\substack{z \in \mathbb{R}^k \\ \|z\| \leq r}} \|y - AG(z)\| + \epsilon$$

Then, with probability $1 - e^{-\Omega(m)}$

$$\|G(\hat{z}) - x^*\| \leq \min_{z \in B^k(r)} \|G(z) - x^*\| + 3\|\eta\| + 2\epsilon + 2\delta$$

Proof: From Lemma 4.1, we know that if $m = \Omega\left(k \log(Lr/\delta)/\alpha^2\right)$ A satisfies S-REC$(G(B^k(r)), 1-\alpha, \delta)$ w.p $1 - e^{-\Omega(\alpha^2 m)}$. Now, applying lemma 4.3 with $\hat{x} = G(\hat{z})$, $S = G(B^k(r))$

$$\|G(\hat{z}) - x^*\| \leq \left[\min_{z \in B^k(r)} \|G(z) - x^*\|\right]\left(\frac{4}{\gamma} + 1\right) + \frac{1}{\gamma}\left(2\|\eta\| + \epsilon + \delta\right)$$

Choose $\alpha = 1/5$, $\gamma = 1 - \alpha = 4/5$

$$\|G(\hat{z}) - x^*\| \leq 6 \min_{z \in B^k(r)} \|G(z) - x^*\| + 2.5\|\eta\| + 1.2\epsilon + 1.2\delta$$

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**Lemma 8.1**: Given $S \subseteq \mathbb{R}^n$, $y \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $\gamma, \delta, \epsilon_1, \epsilon_2 > 0$, if $A$ satisfies S-REC$(S, \gamma, \delta)$ then for any $x_1, x_2 \in S$ s.t $\|Ax_1 - y\| \leq \epsilon_1$ and $\|Ax_2 - y\| \leq \epsilon_2$,

$$\|x_1 - x_2\| \leq (\epsilon_1 + \epsilon_2 + \delta)/\gamma$$

Proof, by S-REC,

$$\|x_1 - x_2\| \leq \frac{1}{\gamma}\left(\|A(x_1 - x_2)\| + \delta\right)$$

Can be potentially very large, how to interpret?

$$= \frac{1}{\gamma}\left(\|(Ax_1 - y) - (Ax_2 - y)\| + \delta\right)$$

$$\leq \frac{1}{\gamma}\left(\|Ax_1 - y\| + \|Ax_2 - y\| + \delta\right) \leq \frac{\epsilon_1 + \epsilon_2 + \delta}{\gamma}$$

**Proof of Lemma 4.1**:

**Lemma 8.2**: Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be an L-Lipschitz function. Let $B^k(r)$ - $L_2$ ball, radius $r$ in $\mathbb{R}^k$; $S = G(B^k(r))$ and $M$ be a $(\delta/L)$-net on $B^k(r)$ s.t $|M| \leq k \log\left(\frac{4Lr}{\delta}\right)$. Let $A \in \mathbb{R}^{m \times n}$ be random, gaussian, $A_{ij} \overset{iid}{\sim} N(0, 1/m)$. if $m = \Omega(k \log(Lr/\delta))$, then for any $x \in S$, if

$$x' = \arg\min_{\hat{x} \in G(M)} \|x - \hat{x}\|, \text{ then}$$

$$\|A(x - x')\| = O(\delta) \quad \text{w.p} \quad 1 - e^{-\Omega(m)}.$$

Lemma A: Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be $L$-Lipschitz. Let

$$B^k(r) = \{z \mid z \in \mathbb{R}^k, \|z\| \le r\}$$

For $\alpha < 1$, if $m = \Omega\left(k/\alpha^2 \log(Lr/\delta)\right)$, then a random matrix $(A_{ij})_{m \times n} \overset{iid}{\sim} N(0, 1/m)$ satisfies

$S$-REC $(G(B^k(r)), 1-\alpha, \delta)$ w.p. $1 - e^{-\Omega(m)}$.

Proof: Construct an $(\delta/L)$-net on $B^k(r)$. There exists a net such that

$$\log |N| \le k \log\left(\frac{4Lr}{\delta}\right)$$

follows by geometric argument. and $(\delta/L) \le 2r$

Now, since $N$ is a $(\delta/L)$ cover of $B^k(r)$ because of Lipschitz, $G(N)$ is a $\delta$-cover of $G(B^k(r))$

Let $T$ denote pairwise differences in $G(N)$ i.e.,

$$T = \{G(z_1) - G(z_2) \mid z_1, z_2 \in N\}$$

$$\log |T| \le 2k \log\left(\frac{4Lr}{\delta}\right)$$

for any $z, z' \in B^k(r)$

$$\begin{aligned} \|G(z) - G(z')\| &\le \|G(z) - G(z_1)\| \\ &+ \|G(z') - G(z_2)\| \\ &+ \underbrace{\|G(z_1) - G(z_2)\|}_{} \le \quad + 2\delta \end{aligned}$$

also, $\|AG_{1}(z_1) - AG_1(z_2)\| \leq \|A[G_1(z) - G_1(z_1)]\|$

$+ \|A[G_1(z') - G_1(z_2)]\|$

$+ \|A[G_1(z) - G_1(z')]\|$

$\leq \|AG_1(z) - AG_1(z')\| + O(\delta)$ come from lemma 8.2

JL :

$P(\|Ax\|^2 \geq (1-\alpha)\|x\|^2, \forall x \in T) \geq 1 - e^{-\Omega(\alpha^2 m)}$

$(1-\alpha)\|G(z_1) - G(z_2)\| \leq \sqrt{1-\alpha}\ \|G(z_1) - G(z_2)\|$

$\leq \|AG(z_1) - AG(z_2)\|$ w.p. $1 - e^{-\Omega(\alpha^2 m)}$

$(1-\alpha)\|G(z) - G(z')\| \leq (1-\alpha)\|G(z_1) - G(z_2)\| + \underbrace{(1+\alpha)O(\delta)}_{O(\delta)}$

$\leq \|A[G(z_1) - G(z_2)]\| + O(\delta)$

$\leq \|A[G(z) - G(z')]\| + O(\delta)$

$\Rightarrow \|AG(z) - AG(z')\| \geq (1-\alpha)\|G(z) - G(z')\| - O(\delta)$

w.p. $1 - e^{-\Omega(\alpha^2 m)}$ for any $z, z' \in B^k(r)$

$\Rightarrow$ A satisfies S-REC $(B^k(r), 1-\alpha, \delta)$ w.p. $1 - e^{-\Omega(\alpha^2 m)}$

$\boxtimes$

Lemma 4.3 (a) Let $A \in \mathbb{R}^{m \times n}$ satisfy S-REC$(S, \gamma, \delta)$ w.p. $1-p$ and

(b) for every $x \in \mathbb{R}^n$, $\|Ax\| \leq 2\|x\|$ w.p. $1-p$. Further, for any $x^* \in \mathbb{R}^n$ and noise $\eta$, $y := Ax^* + n$ and

$$\|y - A\hat{x}\| \leq \min_{x \in S} \|Ax - y\| + \epsilon$$

Then, w.p. $1-2p$,

$$\|\hat{x} - x^*\| \leq \left(\frac{4}{\gamma} + 1\right) \min_{x \in S} \|x^* - x\| + \frac{1}{\gamma}\left(2\|n\| + \epsilon + \delta\right)$$

Proof: Let $\bar{x} = \arg\min_{x \in S} \|x^* - x\|$. Then

$$\|\hat{x} - \bar{x}\| \leq \frac{1}{\gamma}\left(\|A\bar{x} - y\| + \|A\hat{x} - y\| + \delta\right)$$

$$\leq \frac{1}{\gamma}\left(2\|A\bar{x} - y\| + \epsilon + \delta\right)$$

$$\leq \frac{1}{\gamma}\left(2\|A\bar{x} - Ax^*\| + 2\|n\| + \epsilon + \delta\right)$$

$$\leq \frac{1}{\gamma}\left(4\|\bar{x} - x^*\| + 2\|n\| + \epsilon + \delta\right)$$

and $\|\hat{x} - x^*\| \leq \|\bar{x} - x^*\| + \|\hat{x} - \bar{x}\|$

$$\leq \left(\frac{4}{\gamma} + 1\right)\|\bar{x} - x^*\| + \frac{1}{\gamma}\left(2\|n\| + \epsilon + \delta\right)$$

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Proof of lemma 8.2: $\frac{\|Ax\|^2}{\|x\|^2}$ is a sum of $m$ $\chi^2$ r.v's

$\Rightarrow P\left(\|Ax\|^2 \geq (1+\epsilon)^2\|x\|^2\right) \leq 2\exp(-m\epsilon^2)$

so to ensure $P\left(\|Ax\| \geq (1+\epsilon)\|x\|\right) \leq f$, we need

$$\epsilon \geq \sqrt{\frac{1}{m}\log\frac{2}{f}}$$

done differently - used "sub-gamma", constant factors + $\max(\cdot,(\cdot)^2)$ mismatch.

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- $M = M_0 \subseteq M_1 \subseteq \dots \subseteq M_L$ be $\epsilon$-nets of $B^k(r)$
- s.t $M_i$ is $(\delta_i/L)$-net, $\delta_i = \delta_0/2^i$, $\delta_0 = \delta$.

$\exists$ nets s.t $\log|M_i| \leq k\log\left(\frac{4Lr}{\delta_i}\right)$

$$\leq ik + k\log\left(\frac{4Lr}{\delta_0}\right)$$

let $N_i = G(M_i) \Rightarrow N_i$ form $\delta_i$-nets of $G(B^k(r))$
and $|N_i| = |M_i|$

$$T_i \triangleq \{x_{i+1} - x_i \mid x_{i+1} \in N_{i+1}, x_i \in N_i\} \quad i = 0, \dots L-1$$

$$\Rightarrow |T_i| \leq |N_{i+1}|\,|N_i|$$

$$\log|T_i| \leq (2i+1)k + 2k\log\left(\frac{4Lr}{\delta_0}\right)$$

$$\leq 3ik + 2k\log\left(\frac{4Lr}{\delta_0}\right)$$

let $m = 3k \log\left(\frac{4Lr}{\delta_0}\right)$

and $\log \frac{1}{f_i} = m + 4ik$

$$\epsilon_i = 2 + \frac{4}{m} \log \frac{2}{f_i}$$

$$= 2 + \frac{4}{m} \log 2 + 4 + \frac{16ik}{m}$$

$$= O(1) + \frac{16ik}{m}$$

$\forall t \in T_i$, $i = 0, \ldots l-1$

$$P(\|At\| > (1+\epsilon_i)\|t\|) \leq f_i$$

$$\Rightarrow \quad P(\|At\| \leq (1+\epsilon_i)\|t\|, \forall i, \forall t \in T_i)$$

$$\geq 1 - \sum_{i=0}^{l-1} |T_i| f_i$$

$$\log(|T_i| f_i) = \underline{\log |T_i|} + \log f_i$$

$$\leq 3ik + 2k \log\left(\frac{4Lr}{\delta_0}\right) - m - 4ik$$

$$\leq -k\log\left(\frac{4Lr}{\delta_0}\right) - ik$$

$$= -m/3 - ik$$

$$\sum_{j=0}^{l-1} |T_j| f_j \le e^{-m/3} \sum_{j=0}^{l-1} e^{-jk} \le e^{-m/3} \sum_{j=0}^{\infty} e^{-jk} = \frac{e^{-m/3}}{1-1/e}$$

$$\le 2e^{-m/3}$$

Now, for any $x \in S$

$$x = x_0 + (x_1 - x_0) + (x_2 - x_1) \cdots + (x_l - x_{l-1}) + \overbrace{(x - x_l)}^{x_f^{+}}$$

$$x - x_0 = \sum_{j=0}^{l-1} (x_{j+1} - x_j) + x_f$$

where $x_j \in N_j$, $\Rightarrow$ $x_{j+1} - x_j \in T_j$ $\Rightarrow$ wp at least $1 - 2e^{-m/3}$

$$\sum_{j=0}^{l-1} \|A(x_{j+1} - x_j)\| \le \sum_{j=0}^{l-1} (1+\epsilon_j) \|x_{j+1} - x_j\|$$

$$\le \sum (1+\epsilon_j) \delta_j = \delta_0 \sum_{j=0}^{l-1} 2^{-j} \left(O(1) + \frac{16jk}{m}\right)$$

$$= O(\delta_0) + \frac{16\delta_0 k}{m} \sum (j/2^j)$$

$$= O(\delta_0)$$

$$\|x^{+}\| = \|x - x_l\| \le \delta_l = \delta_0 / 2^l, \quad \|x_{j+1} - x_j\| \le \delta_j$$

$$\|A\| \le 2 + \sqrt{\frac{n}{m}} \text{ wp at } 1 - 2e^{-m/2}$$

$$\|A\| \|x^{+}\| \le \left(2 + \sqrt{\frac{n}{m}}\right) \frac{\delta_0}{n} \le \delta_0 \left(\frac{2}{m} + \sqrt{\frac{1}{nm}}\right)$$

$$= \underline{O(\delta_0)} \text{ wp } 1 - 2e^{-m/2}$$

Let $x' = x_0$. $\Rightarrow$ w.p. $1-e^{-\Omega(m)}$

$\|A(x-x')\| = \|A(x-x_0)\|$

$= O(\delta)$

Lemma 8.3: Consider $c$ different $(k-1)$-dim hyperplanes in $\mathbb{R}^k$. # $k$-faces $= O(c^k)$

Proof: $f(c,1) = c+1 = O(c)$

Let $f(c,k-1) = O(c^{k-1})$

$$f(c,k) = \cancel{f(c,k-1)}\ f(c-1,k) + f(c-1,k-1)$$

$$= \underbrace{f(c-1,k)} + O(c^{k-1})$$

$$= f(c-2,k) + f(c-2,k-1) + O(c^{k-1})$$

$$= f(c-2,k) + O(c^{k-1}) + O(c^{k-1})$$

$$\vdots$$

$$= O(c^k)$$

Lemma ~~4.2~~: Let $G: \mathbb{R}^k \to \mathbb{R}^n$ be a $d$-layer NN, each layer is a linear transformation followed by "pointwise non-linearity". Let there be at most $c$-nodes per layer, $m \geq O((kd\log c)/\alpha^2)$, $\alpha < 1$. Then $A_{ij} \overset{iid}{\sim} N(0, 1/m)$ satisfies S-REC$(G(\mathbb{R}^k), 1-\alpha, 0)$ w.p. $1-e^{-\Omega(\alpha^2 m)}$